

Non-orientable quasi-trees for the Bollobás-Riordan polynomial

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Abstract

We extend the quasi-tree expansion of A. Champanerkar, I. Kofman, and N. Stoltzfus to not necessarily orientable ribbon graphs. We study the duality properties of the Bollobás-Riordan polynomial in terms of this expansion. As a corollary, we get a “connected state” expansion of the Kauffman bracket of virtual link diagrams. Our proofs use extensively the partial duality of S. Chmutov.

Keywords: ribbon graph, quasi-tree, partial duality, Bollobás-Riordan polynomial, Kauffman bracket.

1 Introduction

Ribbon graphs are a topological generalization of graphs. They can be described in (at least) three different ways: as embedded graphs, as possibly non-orientable surfaces with boundary or as triples of permutations describing the vertices, the edges and their possible twists (see fig. 1(a)). In the following, we will mainly adopt the surface point of view.

In 1954, W. Tutte defined a graph invariant [17], now named Tutte polynomial, which is a generalization of many other invariants such as the chromatic and flow polynomials. The Tutte polynomial may be described either via a spanning subgraph expansion, a spanning tree expansion, or, recursively, by reduction relations. More recently, B. Bollobás and O. Riordan defined a ribbon graph invariant which generalizes the Tutte polynomial. The Bollobás-Riordan polynomial also has three different possible definitions. The present article focuses on one of them.

It turns out that, for ribbon graphs, the right topological generalization of a spanning tree is a quasi-tree. A quasi-tree is a spanning subribbon graph with only one boundary component (or face). A. Champanerkar, I. Kofman, and N. Stoltzfus proved that the Bollobás-Riordan polynomial has a quasi-tree expansion [2]. Their work was restricted to orientable ribbon graphs. Our article aims at extending their expansion to the non-orientable case.

Very recently, S. Chmutov defined a generalization of the usual Euler-Poincaré (hereafter natural) duality for ribbon graphs [3]. His partial duality consists in forming the natural dual but only with respect to a spanning sub(ribbon)graph. We find that this new duality is an interesting, fruitful and promising framework for the study of ribbon graphs and their invariants. In our opinion, the use of the partial duality simplifies the

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formulation of the proofs presented in this article a lot.

The paper is organized as follows. In section 2, we recall the basic definitions of a ribbon graph and the partial duality. The spanning tree expansion of the Tutte polynomial relies on a notion of activity of an edge with respect to a spanning tree. Section 3 defines the generalization of Tutte’s activities to adapt them to non-plane ribbon graphs and quasi-trees. The spanning tree expansion of the Tutte polynomial consists in a factorization of the monomials of the spanning subgraph expansion. To this end, the subgraphs are grouped into packets, each of which is labelled by a spanning tree. In section 4, we group the subribbon graph of a ribbon graph into packets, naturally associated with quasi-trees. Section 5 is devoted to the statement and proof of our main theorem, namely a quasi-tree expansion of the Bollobás-Riordan polynomial of not necessarily orientable ribbon graphs. We also give the corresponding expansion for the multivariate version of this polynomial [12, 18]. In section 6, we recover the duality property of the Bollobás-Riordan polynomial, namely its invariance under partial duality at $q := xyz^2 = 1$ [3, 18], but in terms of its quasi-tree expansion. This allows us to get an alternative expression for this polynomial at $q = 1$.

The Kauffman bracket of a virtual link diagram and the Bollobás-Riordan polynomial of ribbon graphs have been proven to be related to each other [3–5, 7]. As a consequence, the quasi-tree expansion of the Bollobás-Riordan polynomial allows to get such an expansion for the Kauffman bracket. In section 7, we translate this expansion into pure “knot theoretical” terms to get a connected state (ie a one-component state) expansion of the Kauffman bracket of a virtual link diagram. Finally, an appendix exemplifies the quasi-tree (resp. connected state) expansion of the Bollobás-Riordan polynomial (resp. Kauffman bracket).

Note. *During the publishing process, the author discovered that a quasi-tree expansion for the (unsigned) Bollobás-Riordan polynomial of non-orientable ribbon graphs has been derived by Ed Dewey [8]. His expansion is true for any w but does not make use of Chmutov’s partial duality.*

2 Partial duality of a ribbon graph

2.1 Ribbon graphs

A ribbon graph G is a (not necessarily orientable) surface with boundary represented as the union of two sets of closed topological discs called vertices $V(G)$ and edges $E(G)$. These sets satisfy the following:

- vertices and edges intersect by disjoint line segment,
- each such line segment lies on the boundary of precisely one vertex and one edge,
- every edge contains exactly two such line segments.

Figure 1(a) shows an example of a ribbon graph. Note that we allow the edges to twist (giving the possibility for the surfaces associated with the ribbon graphs to be

non-orientable). A priori an edge may twist more than once but the Bollobás-Riordan polynomial only depends on the parity of the number of twists (this is indeed the relevant information for counting the boundary components of a ribbon graph), so that we will only consider edges with at most one twist.

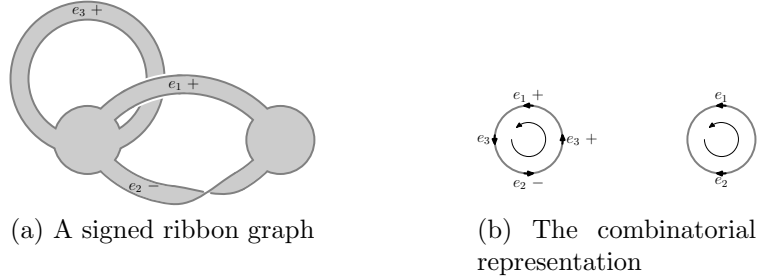


Figure 1: Two representations of a ribbon graph

Definition 2.1 (Notations). *Let G be a ribbon graph. In the rest of this article, we will use the following notation:*

- $v(G) = \text{card } V(G)$ is the number of vertices of G ,
- $e(G) = \text{card } E(G)$ is the number of edges of G ,
- $k(G)$ is its number of components,
- $r(G) = v(G) - k(G)$ is its rank,
- $n(G) = e(G) - r(G)$ is its nullity,
- $f(G)$ is its number of boundary components (faces),
- for all $E' \subseteq E(G)$, $F_{E'}$ is the spanning sub(ribbon) graph of G the edge-set of which is E' and
- for all $E' \subseteq E(G)$, $E'^c := E(G) \setminus E'$.

For the construction of partial dual graphs, another (equivalent) representation of ribbon graphs will be useful. It has been introduced in [3] and will be referred to hereafter as the “combinatorial representation”. It can be described as follows: for any ribbon graph G , pick out an orientation of each vertex-disc and each edge-disc. The orientation of the edges induces an orientation of the line segments along which they intersect the vertices. Then draw all vertex-discs as disjoint circles in the plane oriented counterclockwise (say) but for the edges, draw only the arrows corresponding to the orientation of the line segments. Figure 1(b) gives the combinatorial representation of the graph of fig. 1(a). Each edge $e \in E(G)$ is represented as a pair of arrows which share the same label e .

Given a combinatorial representation, one reconstructs the corresponding ribbon graph as follows. Each circle of the representation is filled: this gives the vertex-discs. Let us consider a couple c_e of arrows with the same label (i.e. corresponding to the same edge). These two arrows belong to the boundaries of vertices v_1 and v_2 , which may be equal. One

draws an edge which intersects v_1 and v_2 along the arrows of c_e . We now have to decide whether this edge twists or not. This depends on the relative direction of the two arrows. Actually there is a unique choice (twist or not) such that there exists an orientation of the edge which reproduces the couple of arrows under consideration. So we proceed as explained for each couple of arrows with a common label.

Loops Unlike the graphs, the ribbon graphs may contain four different kinds of loops. A loop may be **orientable** or not, a **non-orientable** loop being a twisting edge. Let us consider the general situations of fig. 2. The boxes A and B represent any ribbon graph such that the picture 2(a) (resp. 2(b)) describes any ribbon graph G with an orientable (resp. non-orientable) loop e at vertex v . A loop is said to be **nontrivial** if there is a path in G from A to B which does not contain v . If not the loop is called **trivial** [1].

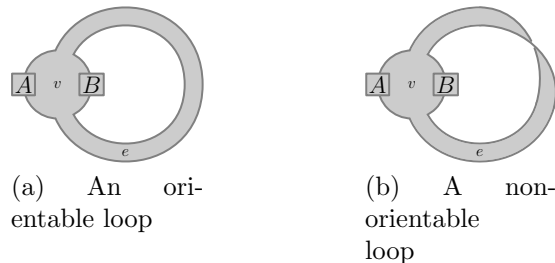


Figure 2: Loops in ribbon graphs

A ribbon graph G is said to be **signed** if an element of $\{+, -\}$ is assigned to each edge. This is achieved via a function $\varepsilon_G : E(G) \rightarrow \{-1, 1\}$.

2.2 Partial duality

S. Chmutov introduced a new “generalized duality” for ribbon graphs which generalizes the usual notion of duality (see [3]). In [13], I. Moffatt renamed this new duality as “partial duality”. We adopt this designation here. We now describe the construction of a partial dual graph and give a few properties of the partial duality.

Let G be a ribbon graph and $E' \subseteq E(G)$. Let $F_{E'}$ be the spanning subribbon graph of G whose edge-set is E' . We will construct the dual $G^{E'}$ of G with respect to the edge-set E' ; see fig. 3(a) for an example. The general idea is the following. We consider the spanning subribbon graph $F_{E'}$ and mark it with arrows to keep track of the edges in $E(G) \setminus E'$. Then we take the natural dual $F_{E'}^*$ of the arrow-marked ribbon graph $F_{E'}$. Finally we use the arrows on $F_{E'}^*$ to redraw the edges in $E(G) \setminus E'$ [14].

We now describe the partial duality more precisely. Recall that each edge of G intersects one or two vertex-discs along two line segments. In the following, each time we write “line segment”, we mean the intersection of an edge and a vertex.

We actually construct the combinatorial representation of the partial dual $G^{E'}$ of G . We first choose an orientation for each edge of G . It induces an orientation of the boundaries of the edges. For each edge in $E(G) \setminus E'$, and as was explained for the combinatorial

representation, we draw one arrow per oriented line segment at the boundary of that edge and in the direction of the orientation. For the edges in E' , we proceed differently. Considering them as rectangles, they have two opposite sides that they share with one or two disc-vertices: these are the line segments defined above. But they also have two other opposite sides that we call “long sides”. The chosen orientation induces an orientation of the long sides of the edges in E' ; see fig. 3(c) for an example. We draw an arrow on each long side of each edge in E' according to the chosen orientation. Now draw each boundary component of $F_{E'}$ as a circle with arrows corresponding to the edges of G . The result is the combinatorial representation of $G^{E'}$; see figs. 3(d) and 3(e). Note that G and $G^{E'}$ are generally embedded into different surfaces (they may have different genera).

As in the case of the natural duality, and for any $E' \subseteq E(G)$, there is a bijection between the edges of G and the edges of its partial dual $G^{E'}$. Let $\phi : E(G) \rightarrow E(G^{E'})$ denote this bijection. We explain now how it is defined from the construction of the partial dual graph. As explained above, on each edge $e \in E(G)$, we draw two arrows compatible with an arbitrarily chosen orientation of this edge. If $e \in E'$, these arrows are drawn on the long sides of e . If $e \in E(G) \setminus E'$, they belong to the line segments along which e intersects its end-vertices. Anyway we label this couple of arrows with $\phi(e)$. Proceeding like that for all edges of G , we build the combinatorial representation of the dual $G^{E'}$ namely we get one circle per boundary component of the spanning sub-ribbon graph $F_{E'}$ of G . On each of these circles, there are arrows which represent the edges of $G^{E'}$. For each couple $c_{e'}$ of arrows that is for each edge e' of $G^{E'}$, there exists a unique $e \in E(G)$ such that $c_{e'}$ bears the label $\phi(e)$. The map ϕ is then clearly a bijection.

For signed graphs, the partial duality comes with a change of the sign function. The function $\varepsilon_{G^{E'}}$ is defined by the following equations: for all $e \in E \setminus E'$, $\varepsilon_{G^{E'}}(e) = \varepsilon_G(e)$ and for all $e \in E'$, $\varepsilon_{G^{E'}}(e) = -\varepsilon_G(e)$.

S. Chmutov proved among other things the following basic properties of the partial duality:

Lemma 2.1 ([3]) *For any ribbon graph G and any subset of edges $E' \subseteq E(G)$, we have*

- $(G^{E'})^{E'} = G$,
- $G^{E(G)} = G^*$ and
- let $e \notin E'$; then $G^{E' \cup \{e\}} = (G^{E'})^{\{e\}}$.

The partial duality allows an interesting and fruitful definition of the contraction of an edge:

Definition 2.1 (Contraction of an edge [1]).

Let G be a ribbon graph and $e \in E(G)$ any of its edges. We define the contraction of e by

$$G/e := G^{\{e\}} - e. \quad (1)$$

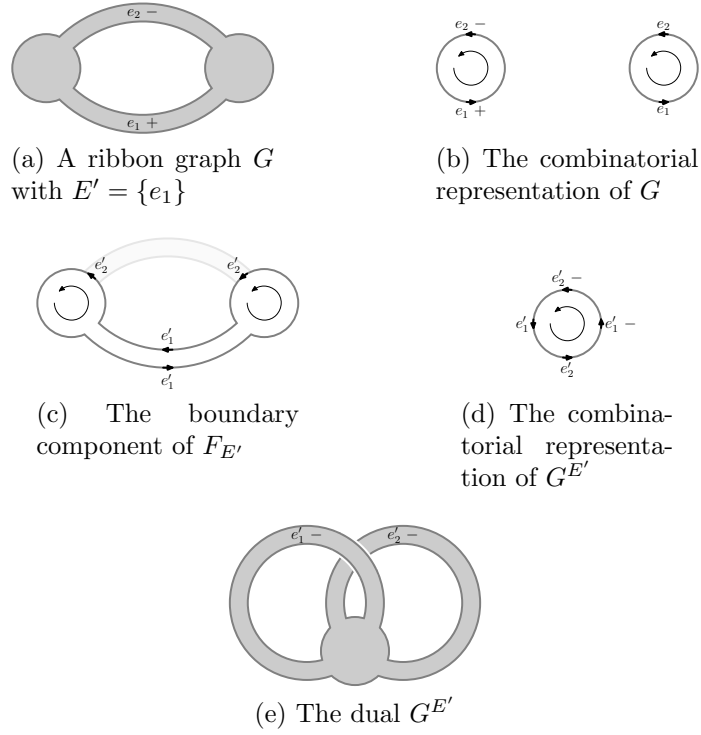


Figure 3: Construction of a partial dual

From the definition of the partial duality, one easily checks that, for an edge incident with two different vertices, the definition 2.1 coincides with the usual intuitive contraction of an edge. The contraction of a loop depends on its orientability; see figs. 4 and 5.

Different definitions of the contraction of a loop have been used in the literature. One can define $G/e := G - e$. In [10], S. Huggett and I. Moffatt give a definition which leads to surfaces which are no longer ribbon graphs. The definition 2.1 maintains the duality between contraction and deletion.

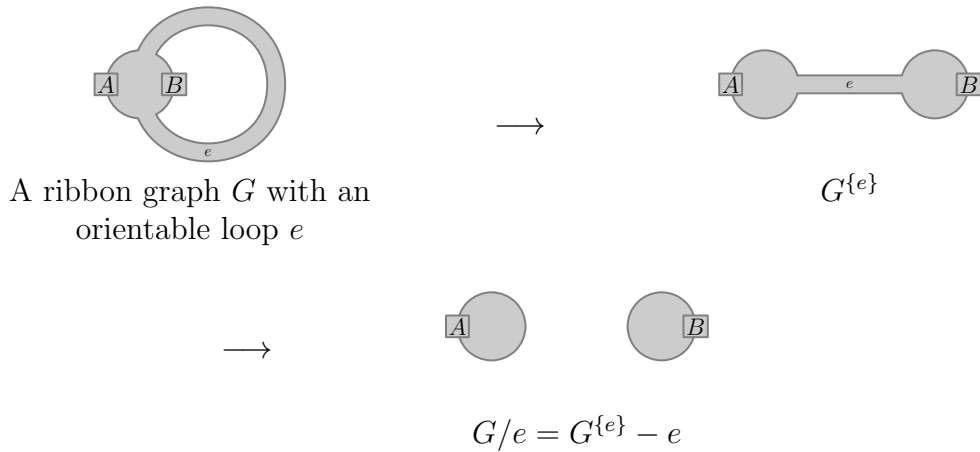


Figure 4: Contraction of an orientable loop

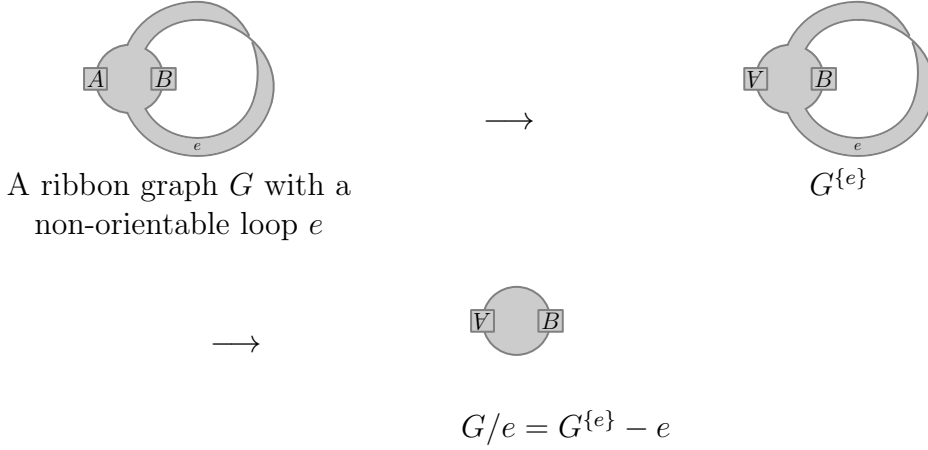


Figure 5: Contraction of a non-orientable loop

3 Activities with respect to a quasi-tree

Definition 3.1 (Quasi-tree [2, 6]). A quasi-tree Q is a ribbon graph with $f(Q) = 1$. Let G be a ribbon graph that is not necessarily orientable. The set of spanning subribbon graphs of G which are quasi-trees is denoted by \mathcal{Q}_G .

A quasi-tree is a generalization of a spanning tree in the following sense. If G is a plane ribbon graph, then \mathcal{Q}_G is the set of spanning trees of G . For a non-plane ribbon graph G , \mathcal{Q}_G contains the spanning trees of G and each quasi-tree contains a spanning tree.

Definition 3.2 (Crossing edges). Let G be a one-vertex ribbon graph. Let $e, e' \in E(G)$ be two edges of G . They intersect the vertex of G along line segments $s_1(e), s_2(e), s_1(e')$ and $s_2(e')$. The edges e and e' **cross** each other (written $e \bowtie e'$) if, turning around the vertex of G (in any direction), one meets the line segments of e and e' alternately, say $s_1(e), s_1(e'), s_2(e), s_2(e')$.

For example, in fig. 6(a), $e_1 \bowtie e_2$, $e_1 \bowtie e_3$ but e_2 and e_3 do not cross each other.

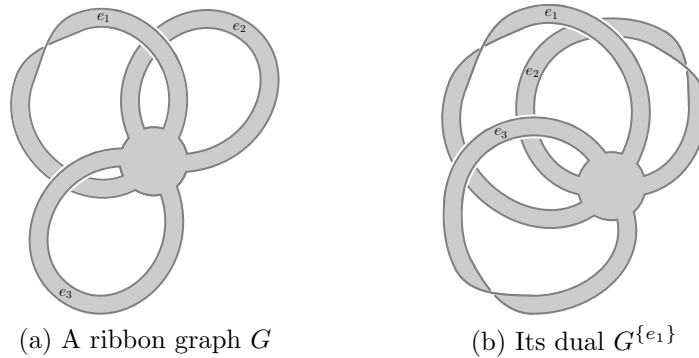


Figure 6: Crossing and linking edges

If Q is a quasi-tree of a ribbon graph G , the partial dual $G^{E(Q)}$ of G is a one-vertex ribbon graph.

Definition 3.3 (Linking edges). Let G be a ribbon graph and $e, e' \in E(G)$ be two of its edges. Let Q be a quasi-tree in G . We say that e and e' **link** each other (with respect to Q) if they cross each other in $G^{E(Q)}$.

One of the quasi-trees of the ribbon graph of figure fig. 6(a) is $F_{\{e_1\}}$. The edges e_2 and e_3 link each other with respect to $F_{\{e_1\}}$: they cross each other in $G^{\{e_1\}}$; see fig. 6(b).

Remark. In [2] the authors associated a chord diagram with any ribbon graph G and quasi-tree $Q \in \mathcal{Q}_G$. They defined two edges to link each other if their corresponding chords cross each other. This definition is actually the same as definition 3.3 as the circle of the chord diagram in [2] is the boundary of the unique vertex in $G^{E(Q)}$.

Definition 3.4 (Activities with respect to a quasi-tree). Let G be a ribbon graph and $Q \in \mathcal{Q}_G$ one of its quasi-trees. Let \prec be a total order on the set $E(G)$ of edges of G . An edge $e \in E(G)$ is said to be **live** if it does not link any lower-ordered edge; otherwise it is **dead**. Moreover e is **internal** if it belongs to $E(Q)$ and **external** otherwise.

We let $\mathcal{I}(Q)$ be the set of internally live edges of G (with respect to \prec). Let $\mathcal{I}_o(Q)$ (resp. $\mathcal{I}_n(Q)$) be the set of internally live edges that form orientable (resp. non-orientable^a) loops in $G^{E(Q)}$. Obviously $\mathcal{I}_o(Q) \cap \mathcal{I}_n(Q) = \emptyset$ and $\mathcal{I}(Q) = \mathcal{I}_o(Q) \cup \mathcal{I}_n(Q)$. We define similarly $\mathcal{E}(Q)$, $\mathcal{E}_o(Q)$ and $\mathcal{E}_n(Q)$ for the externally live edges.

Finally we let $\mathcal{D}(Q)$ be the set of internally dead edges of G with respect to Q and \prec .

One easily checks that for plane ribbon graphs, definition 3.4 of live (resp. dead) edges coincides with the definition of active (resp. inactive) edges in the spanning tree expansion of the Tutte polynomial [17]. In contrast, for non-plane ribbon graphs, those definitions are different. First of all there are more quasi-trees than spanning trees but even with respect to a spanning tree the activities are different. Let us once more consider the example of fig. 6(a) with $e_1 \prec e_2 \prec e_3$. The only spanning tree in G is F_\emptyset (and $G^\emptyset = G$). All edges are externally active but $\mathcal{I} = \mathcal{D} = \emptyset$, $\mathcal{E} = \mathcal{E}_n = \{e_1\}$ and e_2, e_3 are externally dead. With respect to the quasi-tree $F_{\{e_1\}}$, we have $\mathcal{I} = \mathcal{I}_n = \{e_1\}$, $\mathcal{D} = \emptyset$ and e_2, e_3 are externally dead.

^aAs $v(G^{E(Q)}) = 1$, any edge of $G^{E(Q)}$ is a loop. In the following, when we write that an edge is orientable (or not) it always means “as a loop in a certain $G^{E(Q)}$ ”.

4 Binary tree of partial resolutions

Following [2], we construct a rooted binary tree which allows us to group the spanning subribbon graphs of a given connected ribbon graph into packets labelled by the quasi-trees of G . The members of these packets are in one-to-one correspondence with the subsets of orientable internally and externally live edges; see lemma 4.4.

4.1 Partial resolutions and duality

In this section, we prove two lemmas about resolutions and quasi-trees. These lemmas will be useful for the proof of lemma 4.4. The proofs below use Chmutov's partial duality.

Definition 4.1 (Resolutions). *Let G be a ribbon graph. A **resolution** s of G is a map from $E(G)$ into $\{0, 1\}$. Each resolution determines a spanning subribbon graph H_s such that $E(H_s) := \{e \in E(G) : s(e) = 1\}$.*

*A **partial resolution** ρ of G is a map from $E(G)$ into $\{0, 1, *\}$. We define H_ρ to be the spanning subribbon graph of G whose edge-set is $\{e \in E(G) : \rho(e) = 1\}$. We let $U(\rho) := \{e \in E(G) : \rho(e) = *\}$ be the set of unresolved edges. Each partial resolution determines a subset of the spanning subribbon graphs of G : $[\rho] := \{\text{resolutions } s \text{ of } G : s(e) = \rho(e) \text{ if } \rho(e) \in \{0, 1\}\}$.*

Let $F \subseteq G$ be a spanning subribbon graph of G . The number of faces of F equals the number of vertices of its natural dual F^* . But in the following it will be necessary to express this number in terms of the partial dual of G with respect to $E(F)$, namely

$$f(F) = v(F^*) = v(G^{E(F)}). \quad (2)$$

Proposition 4.1 *Let G be a ribbon graph and $F, F' \subseteq G$ two spanning subribbon graphs of G . Let $\Delta := \Delta(F, F') = (E(F) \cup E(F')) \setminus (E(F) \cap E(F'))$. Then we have*

$$f(F') = v((G^{E(F)} - \Delta^c)/\Delta). \quad (3)$$

Proof. As in eq. (2), $f(F') = v(G^{E(F')})$. But $G^{E(F')} = (G^{E(F)})^\Delta$ so $f(F') = v((G^{E(F)})^\Delta) = v((G^{E(F)})^\Delta - E(G))$. Using $E(G) = \Delta \cup \Delta^c$ and for any ribbon graph G and any $E', E'' \subseteq E(G)$ such that $E' \cap E'' = \emptyset$, $G^{E'} - E'' = (G - E'')^{E'}$, we have $f(F') = v((G^{E(F)} - \Delta^c)^\Delta - \Delta) = v((G^{E(F)} - \Delta^c)/\Delta)$ by definition 2.1. \square

Lemma 4.2 *Let G be a ribbon graph and s a resolution of G such that H_s is a quasi-tree. Let e be an edge of G , not necessarily in $E(H_s)$. Let s' be defined by*

$$s' = \begin{cases} s & \text{on } E(G) \setminus \{e\}, \\ 1 - s & \text{on } \{e\}. \end{cases} \quad (4)$$

If e is a non-orientable loop in $G^{E(H_s)}$, then $H_{s'}$ is also a quasi-tree.

Proof. We are going to use proposition 4.1 with $F = H_s$ and $F' = H_{s'}$. As $e \in H_s \iff e \notin H_{s'}$, $\Delta = \{e\}$. F being a quasi-tree, $G^{E(F)}$ is a one-vertex ribbon graph and $G^{E(F)} - \Delta^c =: H'$ consists of the unique vertex of $G^{E(F)}$ and the loop e . By proposition 4.1 the number of faces of F' equals the number of vertices of H'/Δ . Proving that F' is a quasi-tree amounts to proving that $H'/\{e\}$ is a one-vertex graph. By assumption e is non-orientable in $G^{E(H_s)}$. It is then non-orientable in H' . Thanks to the definition 2.1, its contraction leads to a one-vertex ribbon graph. \square

Lemma 4.3 *Let G be a ribbon graph and s a resolution of G such that H_s is a quasi-tree. Let e, e' be two edges of G , not necessarily in $E(H_s)$. Let s' be defined by*

$$s' = \begin{cases} s & \text{on } E(G) \setminus \{e, e'\}, \\ 1 - s & \text{on } \{e, e'\}. \end{cases} \quad (5)$$

If e and e' link each other with respect to H_s and at most one of them is a non-orientable loop in $G^{E(H_s)}$, then $H_{s'}$ is also a quasi-tree.

Proof. We distinguish between three cases: 1. $e, e' \in E(H_s)$, 2. neither e nor e' belongs to $E(H_s)$ and 3. $e \in E(H_s)$ and $e' \notin E(H_s)$ (or the converse). We are now going to use proposition 4.1 with $F = H_s$ and $F' = H_{s'}$. In the three cases, $\Delta = \{e, e'\}$. $H_s = F$ being a quasi-tree, $G^{E(F)}$ is a one-vertex ribbon graph. Then $G^{E(F)} - \Delta^c$ consists of the vertex of $G^{E(F)}$ and the two loops e and e' . By assumption these link each other which means that they cross each other in $G^{E(F)}$.

We have to consider two cases: 1. both e and e' are orientable in $G^{E(H_s)}$, 2. one of them is non-orientable, say e and the other one (e') is orientable.

1. The contraction of e gives two vertices linked by a bridge e' . The contraction of e' is a single vertex.
2. The contraction of e leads to a one-vertex ribbon graph with a single *non-orientable* loop e' . The contraction of e' leads to a single vertex and $f(F') = 1$. \square

4.2 Binary tree

Definition 4.2 (Nugatory edges). *Let G be a ribbon graph and ρ one of its partial resolutions. Let $e \in E(G)$ and ρ_0^e (resp. ρ_1^e) be the partial resolution of G obtained from ρ by resolving e to be 0 (resp. 1). The edge e is called **nugatory** if $[\rho_0^e]$ or $[\rho_1^e]$ does not contain any quasi-tree of G .*

For any connected ribbon graph G and any total order on $E(G)$, we now describe the construction of the binary tree $\mathcal{T}(G)$. Each of its nodes is a partial resolution of G . The construction essentially follows [2]. Let the root of $\mathcal{T}(G)$ be the totally unresolved partial resolution of G : for all $e \in E(G)$, $\rho(e) = *$. We resolve edges, in the reverse order (starting with the highest edge), by changing $*$ to 0 or 1. If an edge is nugatory, it is left unresolved and we proceed to the next edge. For a given node ρ in $\mathcal{T}(G)$, if e is not nugatory then the left child is ρ_0^e and the right child is ρ_1^e . We terminate this process at a leaf when all subsequent edges are nugatory or all edges have been resolved.

Let us now give an example of such a binary tree. We consider the ribbon graph of fig. 6(a) with $e_1 \prec e_2 \prec e_3$. The associated binary tree is represented in fig. 7. Each node of the tree is a partial resolution; for instance $*10$ corresponds to $\rho(e_1) = *, \rho(e_2) = 1$ and $\rho(e_3) = 0$.

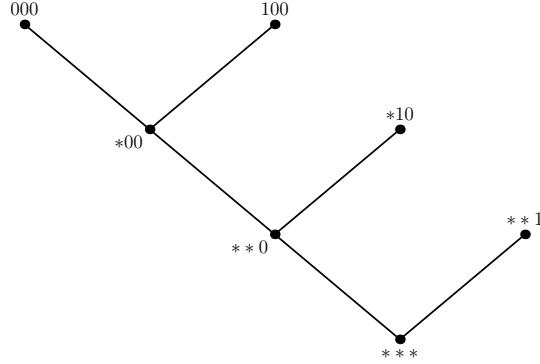


Figure 7: A binary tree of partial resolutions

By construction, each leaf ρ of such a binary tree $\mathcal{T}(G)$ is a partial resolution of G all the unresolved edges of which are nugatory. Therefore there exists a unique resolution $s \in [\rho]$ such that H_s is a quasi-tree. Indeed, let us consider a node of the binary tree $\mathcal{T}(G)$ i.e. a partial resolution σ of G . Let e be the edge to be tested at this node. If e is nugatory, either $[\sigma_0^e]$ or $[\sigma_1^e]$ contains a quasi-tree. If e is not nugatory, they both contain a quasi-tree. Thus, by induction, for each leaf ρ of $\mathcal{T}(G)$, $[\rho]$ contains at least one quasi-tree. Let us assume that it contains more than one quasi-tree. This would mean that there exists an unresolved edge e in ρ such that both $[\rho_0^e]$ and $[\rho_1^e]$ contain a quasi-tree. But this is in contradiction with the fact that all unresolved edges of a leaf are nugatory.

We let Q_ρ be the unique quasi-tree in $[\rho]$. For each spanning subribbon graph H_s , $s \in [\rho]$ we define Q_{H_s} to be Q_ρ .

Lemma 4.4 *Let G be a connected ribbon graph. Let ρ be a leaf of $\mathcal{T}(G)$, and let Q_ρ be the corresponding quasi-tree. If e is unresolved in ρ then e is orientable in $G^{E(Q_\rho)}$ and live with respect to Q_ρ . If e is resolved in ρ , it is either dead with respect to Q_ρ or non-orientable in $G^{E(Q_\rho)}$ and live.*

Proof. Let e be an unresolved edge of a leaf ρ of $\mathcal{T}(G)$. If e is non-orientable in $G^{E(Q_\rho)}$ then by lemma 4.2 there exist two different resolutions in $[\rho]$ corresponding to quasi-trees. This contradicts the fact that e is nugatory. As a conclusion, nugatory edges are orientable in $G^{E(Q_\rho)}$.

Let e_i and e_j be two unresolved edges in ρ , which are therefore nugatory and orientable in $G^{E(Q_\rho)}$. If $e_i \bowtie e_j$, by lemma 4.3, there exists two different resolutions in $[\rho]$ corresponding to quasi-trees. This contradicts the fact that e_i and e_j are nugatory. Thus unresolved edges can only link resolved ones.

Suppose e_i is unresolved in ρ and links a resolved edge e_j with $j \prec i$. Let $s \in [\rho]$ be the resolution such that $H_s = Q_\rho$. The edge e_i being unresolved in ρ , is orientable in

$G^{E(Q_\rho)}$, so we can apply lemma 4.3. Thus there exists another partial resolution s' such that $f(H_{s'}) = 1$. s' is obtained from s by changing only $s(e_i)$ and $s(e_j)$.

Now there exists a unique closest parent $\tilde{\rho}$ of ρ in $\mathcal{T}(G)$ such that e_j is a non-nugatory unresolved edge in $\tilde{\rho}$. If $s \in [\tilde{\rho}_0^{e_j}]$ (say) then $s' \in [\tilde{\rho}_1^{e_j}]$. This implies that e_i is not nugatory in $\tilde{\rho}$ and contradicts the assumption that $j \prec i$ because if that were the case and since edges are resolved in the reverse order, e_i should be nugatory in $\tilde{\rho}$. Thus if $e_i \bowtie e_j$, $i \prec j$ and e_i is live.

Finally, let e_i be a resolved edge in ρ . If e_i links an unresolved edge e_j then by the previous argument $j \prec i$ and e_i is dead. So let us assume that e_i only links resolved edges $\{e_j\}_{j \in R}$, $R \subset \{1, \dots, |E(G)|\}$. If there exists one $j \in R$ such that $j \prec i$, e_i is dead. Suppose therefore that for all $j \in R$, $i \prec j$. There exists a unique closest parent $\tilde{\rho}$ of ρ in $\mathcal{T}(G)$ such that e_i is a non-nugatory unresolved edge in $\tilde{\rho}$. Edges are resolved in reverse order, so the e_j 's, $j \in R$ are resolved in $\tilde{\rho}$. Moreover both $[\tilde{\rho}_0^{e_i}]$ and $[\tilde{\rho}_1^{e_i}]$ contain a quasi-tree. If e_i is orientable and does not link an unresolved edge, it is an orientable trivial loop in $G^{E(Q_\rho)} - \{e_j\}_{j \in R}$. Suppose that $\rho \in [\tilde{\rho}_0^{e_i}]$ (resp. $[\tilde{\rho}_1^{e_i}]$). Then by proposition 4.1, and since Δ and Δ^c being disjoint, we can change the order of contraction and deletion, for all $s \in [\tilde{\rho}_1^{e_i}]$ (resp. $[\tilde{\rho}_0^{e_i}]$), and $f(H_s) = v(G^{E(Q_\rho)}/\Delta - \Delta^c) \geq 2$ with $e_i \in \Delta$ and for all $j \in R$, $e_j \notin \Delta$. Thus either $[\tilde{\rho}_0^{e_i}]$ or $[\tilde{\rho}_1^{e_i}]$ does not contain any quasi-tree which contradicts the fact that e_i is resolved. Therefore e_i links an unresolved edge and is dead. Note finally that if $R = \emptyset$ i.e. if e_i does not link any edge, exactly the same reasoning applies as well. Namely, if $e_i \in [\tilde{\rho}_0^{e_i}]$ (resp. $e_i \in [\tilde{\rho}_1^{e_i}]$), $[\tilde{\rho}_1^{e_i}]$ (resp. $[\tilde{\rho}_0^{e_i}]$) does not contain any quasi-tree. This contradicts the fact that e_i is resolved in ρ and proves that e_i links an unresolved edge. \square

Remark. Concerning the last part of the preceding proof, if e_i is non-orientable and only links higher-ordered edges, it does not need to link an unresolved edge to ensure that both $[\tilde{\rho}_0^{e_i}]$ and $[\tilde{\rho}_1^{e_i}]$ contain a quasi-tree. Thus non-orientable (resolved) edges may be live. For example, in the leaf 100 of the binary tree in fig. 7 (which corresponds to the graph of fig. 6(a)), the edge e_1 is non-orientable, resolved and live.

To sum up this section, we have proven the following

Corollary 4.5 *Let G be a connected ribbon graph and \mathcal{S}_G its set of spanning subribbon graphs. Given a total order on $E(G)$, \mathcal{S}_G is in one-to-one correspondence with $\bigcup_{Q \in \mathcal{Q}_G} \mathcal{I}_o(Q) \times \mathcal{E}_o(Q)$. Namely to each spanning subribbon graph F there corresponds a unique quasi-tree Q_F . Then, there exists $S \subseteq \mathcal{I}_o(Q_F) \cup \mathcal{E}_o(Q_F)$ such that $E(F) = \mathcal{D}(Q_F) \cup \mathcal{I}_n(Q_F) \cup S$.*

5 Non-orientable quasi-tree expansions

5.1 The (signed) Bollobás-Riordan polynomial

This section is devoted to the statement and proof of our main theorem, namely a quasi-tree expansion of the signed Bollobás-Riordan polynomial of not necessarily orientable ribbon graphs. For any subribbon graph F of G , we let $t(F)$ be 0 if F is orientable and 1 otherwise. Recall that for any ribbon graph G , the (unsigned) Bollobás-Riordan polynomial is defined by [1]

$$R(G; x, y, z, w) = \sum_{F \subseteq G} (x-1)^{r(G)-r(F)} y^{n(F)} z^{(k-f+n)(F)} w^{t(F)} \quad (6)$$

considered as an element of the quotient of $\mathbb{Z}[x, y, z, w]$ by the ideal generated by $w^2 - w$.

S. Chmutov and I. Pak introduced an extension of the Bollobás-Riordan polynomial at $w = 1$ [4]. It is a three-variable **polynomial** R_s defined on *signed* ribbon graphs. Recall that a graph is said to be signed if to each of its edges, an element of $\{+, -\}$ is assigned.

For any signed ribbon graph G , let $E_+(G)$ (resp. $E_-(G)$) be the set of positive (resp. negative) edges of G , and let $e_{\pm}(G)$ be their respective cardinalities. For any spanning subribbon graph F of G , let \bar{F} denote the spanning subribbon graph of G with edge-set $E(F)^c$. Let us finally define $s(F) := \frac{1}{2}(e_-(F) - e_-(\bar{F}))$. The signed Bollobás-Riordan polynomial is

$$R_s(G; x+1, y, z) = \sum_{F \subseteq G} x^{k(F)-k(G)+s(F)} y^{n(F)-s(F)} z^{(k-f+n)(F)}. \quad (7)$$

If all the edges of G are positive, $R_s(G; x, y, z) = R(\tilde{G}; x, y, z, 1)$ where \tilde{G} is the underlying unsigned ribbon graph in G .

Before stating our main theorem, we need to recall the definition of the rank polynomial of C. Godsil and G. Royle [9]. It is a four-variable polynomial defined on matroids. Nevertheless, restricting ourselves to graphic matroids, we can easily deduce a version of this polynomial for graphs.

Definition 5.1 (The Rank polynomial [9]). *Let G be a graph (not a ribbon graph). The rank polynomial is defined as follows:*

$$Ra(G; \alpha, \beta, \gamma, \delta) = \sum_{F \subseteq G} \alpha^{e_+(\bar{F})+e_-(F)} \beta^{e_+(F)+e_-(\bar{F})} \gamma^{k(F)-k(G)} \delta^{n(F)} \quad (8)$$

where the sum runs over the spanning subgraphs of G .

Note that the rank polynomial is homogeneous in α, β : the sum of the exponents of α and β is constant and equals $e(G)$. Thus we have

$$Ra(G; \alpha, \beta, \gamma, \delta) = \alpha^{e(G)} Ra(G; 1, \beta/\alpha, \gamma, \delta). \quad (9)$$

The rank polynomial is a generalization of the Tutte polynomial:

$$T(G; x, y) := \sum_{F \subseteq G} (x-1)^{k(F)-k(G)} (y-1)^{n(F)} = Ra(G; 1, 1, x-1, y-1). \quad (10)$$

The signed and unsigned Bollobás-Riordan polynomials are multiplicative on disjoint unions of ribbon graphs, so we can restrict ourselves to connected ribbon graphs, without loss of generality.

Definition 5.2. Let G be a connected ribbon graph. For any total order on $E(G)$ and any quasi-tree $Q \in \mathcal{Q}_G$, let \mathbf{G}_Q be the graph (not the ribbon graph) whose vertices are the components of $F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}$ and whose edges are the internally live orientable edges (namely the elements of $\mathcal{I}_o(Q)$). In other words, consider the graph \tilde{G} underlying G . There is obviously a bijection f between $E(G)$ and $E(\tilde{G})$. Then $G_Q := \tilde{G}/f(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))$ (remember that, in a graph, the contraction of a loop consists in its deletion).

Theorem 5.1 (Quasi-tree expansion) Let G be a connected signed ribbon graph. For any total order on $E(G)$, the signed Bollobás-Riordan polynomial is given by

$$\begin{aligned} R_s(G; x+1, y, z) = & (x^{-1/2}y^{1/2})^{e_-(G)} \sum_{Q \in \mathcal{Q}_G} x^{e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} y^{n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) - e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} \\ & z^{(k-f+n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)})} (1+x)^{e_-(\mathcal{E}_o(Q))} (1+y)^{e_+(\mathcal{E}_o(Q))} \\ & (x^{1/2}y^{-1/2})^{r(G_Q) + e_-(\mathcal{I}_o(Q))} Ra(G_Q; 1, x^{-1/2}y^{1/2}, x^{1/2}y^{1/2}, x^{1/2}y^{1/2}z^2) \end{aligned} \quad (11)$$

where, for all $E' \subseteq E(G)$, $E_{\pm}(E') := E_{\pm}(G) \cap E'$, and $e_{\pm}(E') := |E_{\pm}(G) \cap E'|$.

Corollary 5.2 Let G be a connected ribbon graph. For any total order on $E(G)$, the Bollobás-Riordan polynomial at $w = 1$ is given by

$$R(G; x, y, z, 1) = \sum_{Q \in \mathcal{Q}_G} y^{n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)})} z^{(k-f+n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)})} (1+y)^{|\mathcal{E}_o(Q)|} T(G_Q; x, yz^2 + 1)$$

where $T(G_Q)$ is the Tutte polynomial of G_Q .

Before proving theorem 5.1, let us comment on the fact that, in corollary 5.2, we get a quasi-tree expansion only at $w = 1$. To extend our expansion to the full Bollobás-Riordan polynomial (namely for any w), we would need in particular to relate the orientability of any subgraph to the orientability of $F_{\mathcal{D} \cup \mathcal{I}_n}$. This has been done in [8].

The proof of theorem 5.1 relies on the following lemma:

Lemma 5.3 Let G be a connected ribbon graph. Let $Q \in \mathcal{Q}_G$ be a quasi-tree in G . Given a total order on $E(G)$, and for any $S = S_1 \cup S_2$ with $S_1 \subset \mathcal{I}_o(Q)$ and $S_2 \subset \mathcal{E}_o(Q)$, we have

- $k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) = k(W)$, where W is the spanning subgraph of G_Q , the edge-set of which is S_1 ,
- $f(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = f(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) - |S_1| + |S_2|$.

Proof. The edges in S being orientable, the proof follows the one given in [2]. But we reformulate it in terms of S. Chmutov's duality.

Let $e \in S_2$. We want to prove that $k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S \setminus \{e\}})$ that is to say that e intersects only one component of $F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S \setminus \{e\}}$. Clearly if e intersects only one component of $F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}$, it does so a fortiori in $F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S \setminus \{e\}}$. Then it is enough to prove that $k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1 \cup \{e\}}) = k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1})$. Actually we are going to prove an even stronger statement, namely that e only intersects one boundary component of $F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}$. This would obviously imply the desired result.

The boundary components of a ribbon graph are the vertices of its natural dual. We will therefore prove that e is a loop in $(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1 \cup \{e\}})^{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}$.

$$(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1 \cup \{e\}})^{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1} = G^{E(Q)} / \overline{S_1} - ((\mathcal{E}_o(Q) \setminus \{e\}) \cup \mathcal{E}_n(Q) \cup \mathcal{D}(Q)) \quad (12)$$

$$=: G^{E(Q)} / \overline{S_1} - A \quad (13)$$

with $\overline{S_1} := \mathcal{I}_o(Q) \setminus S_1$, $\mathcal{D}(Q)$ the set of externally dead edges and where we used $E(Q) = \mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup \mathcal{I}_o(Q)$ and definition 2.1. Q being a quasi-tree, $G^{E(Q)} - A$ is a one-vertex ribbon graph with edges in $\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup \mathcal{I}_o(Q) \cup \{e\}$. The edges in $\overline{S_1} \cup \{e\}$ are all unresolved in the partial resolution ρ of $\mathcal{T}(G)$ such that $Q_\rho = Q$. Therefore they do not cross each other in $G^{E(Q)}$; see the proof of lemma 4.4. As a consequence the edge e is still a loop in $G^{E(Q)} / \overline{S_1}$ and the first equality of the first item follows.

The proof that $k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) = k(W)$ is obvious from the definition 5.2 of G_Q .

Let us now prove the second statement of the lemma:

$$f(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = v(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}^*) = v(G^{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = v(G^{E(Q)} / (\overline{S_1} \cup S_2)) \quad (14)$$

$$f(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) = v(G^{E(Q)} / \mathcal{I}_o(Q)) \quad (15)$$

But the edges in $\mathcal{I}_o(Q) \cup \mathcal{E}_o(Q)$ do not cross each other in $G^{E(Q)}$ (see the proof of lemma 4.4). Thus, given the definition 2.1 of the contraction of a loop, we have

$$f(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = v(G^{E(Q)}) + |\overline{S_1}| + |S_2|, \quad (16)$$

$$f(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) = v(G^{E(Q)}) + |S_1| + |\overline{S_1}| \quad (17)$$

which implies $f(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = f(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) - |S_1| + |S_2|$. \square

Corollary 5.4 *Let G be a connected ribbon graph. Let $Q \in \mathcal{Q}_G$ be a quasi-tree in G . Given a total order on $E(G)$ and for any $S = S_1 \cup S_2$ with $S_1 \subset \mathcal{I}_o(Q)$ and $S_2 \subset \mathcal{E}_o(Q)$, we have*

- $n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + n(W) + |S_2|$,
- $(k - f + n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = (k - f + n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + 2n(W)$.

Proof. Using now lemma 5.3, we have

$$n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = (e - v + k)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) \quad (18)$$

$$= e(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) + |S_2| - v(G) + k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) \quad (19)$$

$$= n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) + |S_2|, \quad (20)$$

$$n(W) = e(W) - v(W) + k(W) \quad (21)$$

$$= |S_1| - k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}), \quad (22)$$

$$n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) = e(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + |S_1| - v(G) + k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) \quad (23)$$

$$= e(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) - v(G) + k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + |S_1| - k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) \quad (24)$$

$$= n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + n(W) \quad (25)$$

Equations (eq. (20)) and (eq. (25)) imply $n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + n(W) + |S_2|$.

$$(k - f + n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) = k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) - f(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + |S_1| - |S_2| + n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + n(W) + |S_2| \quad (26)$$

$$= (k - f + n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1}) - k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + |S_1| + n(W) \quad (27)$$

$$= (k - f + n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) + 2n(W) \quad (28)$$

which proves corollary 5.4. \square

Proof of theorem 5.1. Thanks to corollary 4.5, the signed Bollobás-Riordan polynomial can be written as follows

$$R_s(G; x + 1, y, z) = (x^{-1/2} y^{1/2})^{e_-(G)} \sum_{Q \in \mathcal{Q}_G} \sum_{S_1 \subset \mathcal{I}_o(Q)} \sum_{S_2 \subset \mathcal{E}_o(Q)} x^{k(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) - k(G) + e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S)} y^{n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S}) - e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S)} z^{(k - f + n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S})} \quad (29)$$

where $S = S_1 \cup S_2$. Using now lemma 5.3 and corollary 5.4, we have

$$R_s(G; x + 1, y, z) = (x^{-1/2} y^{1/2})^{e_-(G)} \sum_{Q \in \mathcal{Q}_G} x^{e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} y^{n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) - e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} z^{(k - f + n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)})} \sum_{S_2 \subseteq \mathcal{E}_o(Q)} x^{e_-(S_2)} y^{e_+(S_2)} \sum_{W \subseteq G_Q} x^{k(W) - k(G_Q) + e_-(W)} (yz^2)^{n(W)} y^{-e_-(W)} \quad (30)$$

$$= (x^{-1/2} y^{1/2})^{e_-(G)} \sum_{Q \in \mathcal{Q}_G} x^{e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} y^{n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)}) - e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} z^{(k - f + n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)})} (1 + x)^{e_-(\mathcal{E}_o(Q))} (1 + y)^{e_+(\mathcal{E}_o(Q))} \sum_{W \subseteq G_Q} x^{k(W) - k(G_Q) + e_-(W)} (yz^2)^{n(W)} y^{-e_-(W)} \quad (31)$$

where we used $k(G) = k(G_Q) = 1$. To conclude, it remains to prove that

$$\begin{aligned} & \sum_{W \subseteq G_Q} x^{k(W)-k(G_Q)+e_-(W)} (yz^2)^{n(W)} y^{-e_-(W)} \\ &= (x^{1/2} y^{-1/2})^{r(G_Q)+e_-(\mathcal{I}_o(Q))} Ra(G_Q; 1, x^{-1/2} y^{1/2}, x^{1/2} y^{1/2}, x^{1/2} y^{1/2} z^2) \end{aligned} \quad (32)$$

which is easily checked from definition 5.1 of the rank polynomial. \square

corollary 5.2 is a direct consequence of theorem 5.1. It is indeed easily verified that, if G is a signed ribbon graph with only positive edges, the right hand side of (eq. (11)) reduces to the desired expression of corollary 5.2.

5.2 The multivariate Bollobás-Riordan polynomial

Multivariate versions of (ribbon) graph polynomials consist in attaching a different indeterminate to each edge. The multivariate Bollobás-Riordan polynomial is defined as follows [12]: let G be a ribbon graph,

$$Z(G; q, \beta, c) := \sum_{F \subseteq G} q^{k(F)} \left(\prod_{e \in E(F)} \beta_e \right) c^{f(F)} \quad (33)$$

where $\beta = \{\beta_e : e \in E(G)\}$. Let G be a graph; the multivariate Tutte polynomial is defined as [16]

$$Z_T(G; q, \beta) := \sum_{F \subseteq G} q^{k(F)} \left(\prod_{e \in E(F)} \beta_e \right). \quad (34)$$

Lemma 5.5 *Let G be a connected ribbon graph. For any total order on $E(G)$, the multivariate Bollobás-Riordan polynomial Z is given by*

$$Z(G; q, \beta, c) = \sum_{Q \in \mathcal{Q}_G} \left(\prod_{e \in \mathcal{D}(Q) \cup \mathcal{I}_n(Q)} \beta_e \right) c^{f(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} \left(\prod_{e \in \mathcal{E}_o(Q)} (1 + c\beta_e) \right) Z_T(G_Q; q, \beta/c).$$

Proof. Thanks to corollary 4.5, the multivariate Bollobás-Riordan polynomial can be written as follows

$$Z(G; q, \beta, c) = \sum_{Q \in \mathcal{Q}_G} \sum_{S_1 \subset \mathcal{I}_o(Q)} \sum_{S_2 \subset \mathcal{E}_o(Q)} q^{k(\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S)} \left(\prod_{e \in \mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S} \beta_e \right) c^{f(\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S)} \quad (35)$$

where $S = S_1 \cup S_2$. Using now lemma 5.3, we have

$$\begin{aligned} Z(G; q, \beta, c) &= \sum_{Q \in \mathcal{Q}_G} \left(\prod_{e \in \mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S} \beta_e \right) c^{f(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} \sum_{S_2} \left(\prod_{e \in S_2} c\beta_e \right) \\ &\quad \times \sum_{S_1} q^{k(\mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S_1)} \left(\prod_{e \in S_1} \beta_e/c \right) \end{aligned} \quad (36)$$

$$\begin{aligned} &= \sum_{Q \in \mathcal{Q}_G} \left(\prod_{e \in \mathcal{D}(Q) \cup \mathcal{I}_n(Q) \cup S} \beta_e \right) c^{f(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} \left(\prod_{e \in \mathcal{E}_o(Q)} (1 + c\beta_e) \right) \\ &\quad \times \sum_{W \subseteq G_Q} q^{k(W)} \left(\prod_{e \in E(W)} \beta_e/c \right) \end{aligned} \quad (37)$$

and the lemma follows. \square

In [18] a multivariate extension of this signed polynomial has been defined and studied. Its invariance under the partial duality has also been proven in [18]. The multivariate signed Bollobás-Riordan polynomial is defined as follows:

$$Z_s(G; q, \alpha, c) := \sum_{F \subseteq G} q^{k(F)+s(F)} \left(\prod_{\substack{e \in E_+(F) \\ \cup E_-(\bar{F})}} \alpha_e \right) c^{f(F)}. \quad (38)$$

It is a multivariate generalization of R_s . Indeed if for any $e \in E(G)$, $\alpha_e = yz$ and if we let \mathbf{yz} be the corresponding set, we have

$$R_s(G; x+1, y, z) = x^{-k(G)} (yz)^{-v(G)} Z_s(G; xyz^2, \mathbf{yz}, z^{-1}). \quad (39)$$

The multivariate polynomial Z_s is actually related to the (unsigned) multivariate Bollobás-Riordan polynomial by

$$Z_s(G; q, \alpha, c) = \left(\prod_{e \in E_-(G)} q^{-1/2} \alpha_e \right) Z(G; q, \beta, c) \quad (40)$$

$$\text{with } \beta_e = \begin{cases} \alpha_e & \text{if } e \text{ is positive,} \\ q\alpha_e^{-1} & \text{if } e \text{ is negative.} \end{cases} \quad (41)$$

It is then an easy exercise to get a quasi-tree expansion for the signed multivariate Bollobás-Riordan polynomial from lemma 5.5.

6 Duality properties

In this section, we first recover the duality property of the Bollobás-Riordan polynomial, namely its invariance at $q = xyz^2 = 1$ [3, 18], but via its quasi-tree expansion. As a consequence, we get another expression for the Bollobás-Riordan polynomial at $q = 1$.

In [3, 18], it has been proven that, for any signed ribbon graph G and any subset $E' \subseteq E(G)$ of edges,

$$Z_s(G; 1, \alpha, c) = Z_s(G^{E'}; 1, \alpha, c), \quad (42a)$$

$$\text{where } Z_s(G; xyz^2, \mathbf{yz}, z^{-1}) := x^{k(G)} (yz)^{v(G)} R_s(G; x+1, y, z). \quad (42b)$$

To prove equation (eq. (42a)), S. Chmutov first exhibited a bijection between the sub-ribbon graphs of G and those of $G^{E'}$. Let us write \mathcal{S}_G for the set of spanning subribbon graphs of G . The bijection is the following map:

$$\begin{aligned} \varphi : \mathcal{S}_G &\rightarrow \mathcal{S}_{G^{E'}} \\ F &\mapsto F' \text{ s.t. } E(F') = E' \Delta E(F), \end{aligned} \quad (43)$$

where Δ stands for the symmetric difference. Then, defining

$$Z_s(G; q, \alpha, c) =: \sum_{F \in \mathcal{S}_G} M_G(F; q, \alpha, c), \quad (44)$$

he proved that $M_G(F; 1, \alpha, c) = M_{G^{E'}}(\varphi(F); 1, \alpha, c)$.

The quasi-tree expansion (eq. (11)) (or lemma 5.5) is a way to factorize some of the monomials $M_G(F)$, naturally associated with a single quasi-tree Q of G . Defining

$$Z_s(G; q, \alpha, c) =: \sum_{Q \in \mathcal{Q}_G} N_G(Q; q, \alpha, c), \quad (45)$$

each monomial $N_G(Q)$ is the sum of several $M_G(F)$ s. In the following, we prove that the bijection (eq. (43)) also preserves the $N_G(Q)$ s:

Lemma 6.1 *For any signed ribbon graph G and any subset $E' \subseteq E(G)$, $N_G(Q; 1, \alpha, c) = N_{G^{E'}}(\varphi(Q); 1, \alpha, c)$.*

In the following, if P is a rational function in one variable, and for all $A \subseteq E(G)$, we abbreviate $\prod_{e \in A} P(\alpha_e)$ as $\mathbf{P}(\alpha)^A$.

Proof. First, note that, from lemma 5.5 and equation (eq. (40)),

$$N_G(Q; q, \alpha, c) =: (\alpha/\sqrt{q})^{E_-(G)} \alpha^{E_+(\mathcal{D} \cup \mathcal{I}_n(Q))} (q/\alpha)^{E_-(\mathcal{D} \cup \mathcal{I}_n(Q))} c^{f(F_{\mathcal{D} \cup \mathcal{I}_n(Q)})} \\ (1 + \alpha c)^{E_+(\mathcal{E}_o)} (1 + qc/\alpha)^{E_-(\mathcal{E}_o)} Z_R(G_Q; q, \beta/c), \quad (46)$$

where β is given by equation (eq. (41)), so that

$$Z_R(G_Q; q, \beta/c) := \sum_{F \subseteq G_Q} q^{k(F) + e_-(F)} (\alpha/c)^{E_+(F)} (\alpha c)^{-E_-(F)}. \quad (47)$$

For $q = 1$, we can explicitly perform the summation over the spanning subgraphs of G_Q to get

$$N_G(Q; 1, \alpha, c) = \alpha^{E_-(G) + E_+(\mathcal{D} \cup \mathcal{I}_n(Q)) - E_-(\mathcal{D} \cup \mathcal{I}_n(Q)) - E_-(\mathcal{I}_o \cup \mathcal{E}_o(Q))} c^{f(F_{\mathcal{D} \cup \mathcal{I}_n(Q)}) - |\mathcal{I}_o(Q)|} \\ (1 + \alpha c)^{E_+(\mathcal{E}_o(Q)) + E_-(\mathcal{I}_o(Q))} (\alpha + c)^{E_-(\mathcal{E}_o(Q)) + E_+(\mathcal{I}_o(Q))}. \quad (48)$$

To prove the lemma, let us first prove that the bijection φ conserves the number of faces:

$$f(F') = v(F'^*) = v((G^{E'})^{E(F')}) = v((G^{E'})^{E' \Delta E(F)}) = v(G^{E(F)}) = f(F). \quad (49)$$

This implies that, if Q is a quasi-tree of G , then $\varphi(Q)$ is a quasi-tree of $G^{E'}$. Moreover, as defined in section 3, an edge $e \in E(G)$ is live (resp. orientable) with respect to Q if it does not cross any lower-ordered edge (resp. if it is an orientable loop) in $G^{E(Q)}$. Then, an edge $e \in E(G^{E'})$ is live (resp. orientable) with respect to $\varphi(Q)$ if it does not cross any lower-ordered edge (resp. if it is an orientable loop) in $(G^{E'})^{E(\varphi(Q))} = (G^{E'})^{E' \Delta E(Q)} = G^{E(Q)}$. Thus, the sets of orientable (resp. non-orientable) live (and dead) edges with respect to Q in G and with respect to $\varphi(Q)$ in $G^{E'}$ are the same. Nevertheless, as $E(Q)$ and $E(\varphi(Q)) = E' \Delta E(Q)$ are different, some internal edges with respect to Q may be external with respect to $\varphi(Q)$, and vice versa. For example, the (internal) edges of $G^{E'}$ with respect to $F_{\varphi(Q)}$ (i.e. $\varphi(Q)$) contain both internal edges (the ones in $E(Q) \setminus E'$)

and external edges (the ones in $E' \setminus E(Q)$) of G with respect to Q . In other words, having

$$E(G) = (\mathcal{D} \cup \mathcal{I}_o \cup \mathcal{I}_n)(Q) \cup (\mathfrak{D} \cup \mathcal{E}_o \cup \mathcal{E}_n)(Q), \quad (50a)$$

$$E(Q) = (\mathcal{D} \cup \mathcal{I}_o \cup \mathcal{I}_n)(Q) \quad (50b)$$

where $\mathfrak{D}(Q)$ is the set of externally dead edges, we have

$$(\mathcal{D} \cup \mathfrak{D})(Q) = (\mathcal{D} \cup \mathfrak{D})(\varphi(Q)), \quad (51a)$$

$$(\mathcal{I}_o \cup \mathcal{E}_o)(Q) = (\mathcal{I}_o \cup \mathcal{E}_o)(\varphi(Q)), \quad (51b)$$

$$(\mathcal{I}_n \cup \mathcal{E}_n)(Q) = (\mathcal{I}_n \cup \mathcal{E}_n)(\varphi(Q)). \quad (51c)$$

And more precisely,

$$(\mathcal{D} \cup \mathcal{I}_n)(\varphi(Q)) = [(\mathcal{D} \cup \mathcal{I}_n)(Q) \setminus E'] \cup [(\mathfrak{D} \cup \mathcal{E}_n)(Q) \cap E']. \quad (52)$$

Also remember that if G is a signed ribbon graph, and $E' \subseteq E(G)$, for all $e \in E'$, the sign of e in $G^{E'}$ is opposite to the sign of e in G ; see section section 2.2. Thus

$$E_{\pm}[(\mathcal{D} \cup \mathcal{I}_n)(\varphi(Q))] = E_{\pm}[(\mathcal{D} \cup \mathcal{I}_n)(Q) \setminus E'] \cup E_{\mp}[(\mathfrak{D} \cup \mathcal{E}_n)(Q) \cap E']. \quad (53)$$

Similarly,

$$\mathcal{E}_o(\varphi(Q)) = [\mathcal{E}_o(Q) \setminus E'] \cup [\mathcal{I}_o(Q) \cap E'] \quad (54)$$

$$E_{\pm}[\mathcal{E}_o(\varphi(Q))] = E_{\pm}[\mathcal{E}_o(Q) \setminus E'] \cup E_{\mp}[\mathcal{I}_o(Q) \cap E'], \quad (55)$$

$$\mathcal{I}_o(\varphi(Q)) = [\mathcal{I}_o(Q) \setminus E'] \cup [\mathcal{E}_o(Q) \cap E'] \quad (56)$$

$$E_{\pm}[\mathcal{I}_o(\varphi(Q))] = E_{\pm}[\mathcal{I}_o(Q) \setminus E'] \cup E_{\mp}[\mathcal{E}_o(Q) \cap E']. \quad (57)$$

Before concluding our proof, we need to relate the number of faces of $F_{\mathcal{D} \cup \mathcal{I}_n(\varphi(Q))} \in \mathcal{S}_{G^{E'}}$ to the number of faces of $F_{\mathcal{D} \cup \mathcal{I}_n(Q)} \in \mathcal{S}_G$.

$$f(F_{\mathcal{D} \cup \mathcal{I}_n(\varphi(Q))}) = v(F_{\mathcal{D} \cup \mathcal{I}_n(\varphi(Q))}^{\star}) = v((G^{E'})^{\mathcal{D} \cup \mathcal{I}_n(\varphi(Q))}) \quad (58)$$

But $\mathcal{D} \cup \mathcal{I}_n(\varphi(Q)) = (E' \Delta E(Q)) \setminus ((\mathcal{I}_o(Q) \setminus E') \cup (\mathcal{E}_o(Q) \cap E'))$, using $E(Q) = (\mathcal{D} \cup \mathcal{I}_o \cup \mathcal{I}_n)(Q)$, so

$$\begin{aligned} f(F_{\mathcal{D} \cup \mathcal{I}_n(\varphi(Q))}) &= v((G^{E'})^{(E' \Delta E(Q)) \setminus ((\mathcal{I}_o(Q) \setminus E') \cup (\mathcal{E}_o(Q) \cap E'))}) \\ &= v((G^{E(Q)})^{((\mathcal{I}_o(Q) \setminus E') \cup (\mathcal{E}_o(Q) \cap E'))}) = 1 + |\mathcal{I}_o(Q) \setminus E'| + |\mathcal{E}_o(Q) \cap E'|, \end{aligned} \quad (59)$$

thanks to the fact that the edges in $\mathcal{I}_o(Q) \cup \mathcal{E}_o(Q)$ do not cross each other in $G^{E(Q)}$. With the same kind of reasoning, we get

$$f(F_{\mathcal{D} \cup \mathcal{I}_n(Q)}) = v((G^{E(Q)})^{\mathcal{I}_o(Q)}) = 1 + |\mathcal{I}_o(Q)| = 1 + |\mathcal{I}_o(Q) \setminus E'| + |\mathcal{I}_o(Q) \cap E'| \quad (60)$$

and obtain

$$f(F_{\mathcal{D} \cup \mathcal{I}_n(\varphi(Q))}) = f(F_{\mathcal{D} \cup \mathcal{I}_n(Q)}) - |\mathcal{I}_o(Q) \cap E'| + |\mathcal{E}_o(Q) \cap E'|. \quad (61)$$

We are now ready to perform the last computation of this proof. We define $Q' := \varphi(Q)$ and $N_{G^{E'}}(Q'; 1, \alpha, c) =: \alpha^{D_\alpha} c^{d_c} (1 + \alpha c)^{D_1} (\alpha + c)^{D_2}$ with

$$D_\alpha = E_-(G^{E'}) \cup E_+(\mathcal{D} \cup \mathcal{I}_n(Q')) \setminus (E_-(\mathcal{D} \cup \mathcal{I}_n(Q')) \cup E_-(\mathcal{I}_o \cup \mathcal{E}_o(Q'))), \quad (62a)$$

$$d_c = f(F_{\mathcal{D} \cup \mathcal{I}_n(Q')}) - |\mathcal{I}_o(Q')|, \quad (62b)$$

$$D_1 = E_+(\mathcal{E}_o(Q')) \cup E_-(\mathcal{I}_o(Q')), \quad (62c)$$

$$D_2 = E_-(\mathcal{E}_o(Q')) \cup E_+(\mathcal{I}_o(Q')). \quad (62d)$$

Now, using equations (55) and (57),

$$\begin{aligned} D_\alpha = & (E_-(G) \cup E_+(E')) \setminus E_-(E') \cup E_+[(\mathcal{D} \cup \mathcal{I}_n)(Q) \setminus E'] \cup E_-[(\mathcal{D} \cup \mathcal{E}_n)(Q) \cap E'] \\ & \setminus (E_-[(\mathcal{D} \cup \mathcal{I}_n)(Q) \setminus E'] \cup E_+[(\mathcal{D} \cup \mathcal{E}_n)(Q) \cap E']) \\ & \cup E_-[(\mathcal{I}_o \cup \mathcal{E}_o)(Q) \setminus E'] \cup E_+[(\mathcal{I}_o \cup \mathcal{E}_o)(Q) \cap E']. \end{aligned} \quad (63)$$

As $E' = E' \cap E(G) = E' \cap [(\mathcal{D} \cup \mathcal{I}_n \cup \mathcal{I}_o \cup \mathcal{E}_o \cup \mathcal{D} \cup \mathcal{E}_n)(Q)]$, we have $(\mathcal{D} \cup \mathcal{E}_n)(Q) \cap E' = [E' \setminus (\mathcal{D} \cup \mathcal{I}_n)(Q)] \setminus [(\mathcal{I}_o \cup \mathcal{E}_o)(Q) \cap E']$, and

$$\begin{aligned} D_\alpha = & E_-(G) \cup E_+(E') \cup E_+[(\mathcal{D} \cup \mathcal{I}_n)(Q) \setminus E'] \cup E_-[E' \setminus (\mathcal{D} \cup \mathcal{I}_n)(Q)] \\ & \setminus (E_+[E' \setminus (\mathcal{D} \cup \mathcal{I}_n)(Q)] \cup E_-(E') \cup E_-[(\mathcal{I}_o \cup \mathcal{E}_o)(Q) \cap E']) \\ & \cup E_-[(\mathcal{D} \cup \mathcal{I}_n)(Q) \setminus E'] \cup E_-[(\mathcal{I}_o \cup \mathcal{E}_o)(Q) \setminus E'] \end{aligned} \quad (64)$$

$$= E_-(G) \cup E_+(\mathcal{D} \cup \mathcal{I}_n(Q)) \setminus (E_-(\mathcal{D} \cup \mathcal{I}_n(Q)) \cup E_-(\mathcal{I}_o \cup \mathcal{E}_o(Q))). \quad (65)$$

Using eqs. (55) to (57) and (61),

$$d_c = f(F_{\mathcal{D} \cup \mathcal{I}_n(Q)}) - |\mathcal{I}_o(Q) \cap E'| + |\mathcal{E}_o(Q) \cap E'| - |\mathcal{I}_o(Q) \setminus E'| - |\mathcal{E}_o(Q) \cap E'| \quad (66)$$

$$= f(F_{\mathcal{D} \cup \mathcal{I}_n(Q)}) - |\mathcal{I}_o(Q)|, \quad (67)$$

$$D_1 = E_+[\mathcal{E}_o(Q) \setminus E'] \cup E_-[\mathcal{I}_o(Q) \cap E'] \cup E_-[\mathcal{I}_o(Q) \setminus E'] \cup E_+[\mathcal{E}_o(Q) \cap E'] \quad (68)$$

$$= E_+(\mathcal{E}_o(Q)) \cup E_-(\mathcal{I}_o(Q)), \quad (69)$$

$$D_2 = E_-[\mathcal{E}_o(Q) \setminus E'] \cup E_+[\mathcal{I}_o(Q) \cap E'] \cup E_+[\mathcal{I}_o(Q) \setminus E'] \cup E_-[\mathcal{E}_o(Q) \cap E'] \quad (70)$$

$$= E_-(\mathcal{E}_o(Q)) \cup E_+(\mathcal{I}_o(Q)). \quad (71)$$

This proves that $N_G(Q; 1, \alpha, c) = N_{G^{E'}}(\varphi(Q); 1, \alpha, c)$, meaning that the bijection (eq. (43)) conserves independently each of the terms (i.e. the $N(Q)$'s) of the quasi-tree expansion. This implies, of course, $Z(G; 1, \alpha, c) = Z(G^{E'}; 1, \alpha, c)$. \square

The preceding lemma shows that, given a ribbon graph G , a subset of edges E' and a quasi-tree $Q \in \mathcal{Q}_G$, there exists a quasi-tree $Q' \in \mathcal{Q}_{G^{E'}}$ such that $N_G(Q; 1, \alpha, c) = N_{G^{E'}}(Q'; 1, \alpha, c)$. The subribbon graph Q' is such that $E(Q') = \varphi(Q)$. But we can also invert the logic: given a ribbon graph G , a quasi-tree $Q \in \mathcal{Q}_G$ and a subset of edges $A \subseteq E(G)$, there exists a subset E' such that the spanning subribbon graph Q' with the property that $E(Q') = A$ is a quasi-tree in $G^{E'}$ and $N_G(Q; 1, \alpha, c) = N_{G^{E'}}(Q'; 1, \alpha, c)$.

Whatever subset A we choose, the bijection ensures that Q' is a quasi-tree in $G^{E'}$. This means that we can fix A and deduce the set E' . A very simple case is $A = \emptyset$: given $Q \in \mathcal{Q}_G$, in which partial dual of G is the empty set a quasi-tree? The answer is given by the bijection φ :

$$E' \Delta E(Q) = \emptyset \iff E' = E(Q). \quad (72)$$

And we get: for any quasi-tree $Q \in \mathcal{Q}_G$, $N_G(Q; 1, \alpha, c) = N_{G^{E(Q)}}(F_\emptyset; 1, \alpha, c)$. In that case, Q' having no edge, the live (or dead) edges are necessarily external. Let us define $\mathcal{L}_o(Q) := \{\text{orientable live edges of } G^{E(Q)} \text{ with respect to } F_\emptyset\}$.

Lemma 6.2 *For any ribbon graph G , the quasi-tree expansion for Z_s at $q = 1$ can be rewritten as follows:*

$$Z_s(G; 1, \alpha, c) = c \sum_{Q \in \mathcal{Q}_G} \alpha^{e_-(G^{E(Q)})} (1 + \alpha c)^{e_+(\mathcal{L}_o(Q))} (1 + c/\alpha)^{e_-(\mathcal{L}_o(Q))}. \quad (73)$$

7 The Kauffman bracket of a virtual link diagram

In [3], S. Chmutov unified several Thistlethwaite like theorems [4, 5, 7, 11, 15] (that is theorems relating link and (ribbon) graph polynomials). He proved that the Kauffman bracket of a virtual link diagram L equals (an evaluation of) the signed Bollobás-Riordan polynomial of a certain ribbon graph G_L^s ; see eq. (74). The latter is constructed from a state \mathfrak{s} of L ; see below and/or [3]. The equality is true for *any* state \mathfrak{s} .

$$[L](A, B, d) = A^{n(G_L)} B^{r(G_L)} d^{k(G_L)-1} R_s(G_L^s; \frac{Ad}{B} + 1, \frac{Bd}{A}, \frac{1}{d}). \quad (74)$$

The new *partial* duality of S. Chmutov ensures the independence of the right hand side of (74) with respect to the state \mathfrak{s} .

In the previous sections, we obtained a quasi-tree expansion for the Bollobás-Riordan polynomial. Thanks to equation (74), we can obviously get such an expansion for the Kauffman bracket. Nevertheless, this expansion would be expressed in terms of parameters (number of vertices, edges etc) of the (subribbon graphs of the) ribbon graph G_L^s associated with the state \mathfrak{s} of L . Here we would like to get a new expansion for the Kauffman bracket, directly expressed in terms of the parameters of the states of L .

Combining equations (eq. (74)) and (eq. (42b)), we get

$$[L](A, B, d) = A^{e(G_L^s)} d^{-1} Z(G_L^s; 1, B/A, d). \quad (75)$$

Now, using the expansion (eq. (73)),

$$[L](A, B, d) = A^{e(G_L^s)} \sum_{Q \in \mathcal{Q}_{G_L^s}} (B/A)^{e_-(G_L^s)^{E(Q)}} (1 + Bd/A)^{e_+(\mathcal{L}_o(Q))} (1 + Ad/B)^{e_-(\mathcal{L}_o(Q))}. \quad (76)$$

Let us now translate this expression into pure “knot theoretical” terms. For this, we need to recall how the ribbon graph $G_L^{\mathfrak{s}}$ is built, out of the state \mathfrak{s} of the virtual link diagram L . The state \mathfrak{s} consists in a set of (possibly nested) circles, called state circles, which writhe at the virtual crossings; see fig. 9(b) for an example. For each state of L , each classical crossing is resolved i.e. at each classical crossing, one performs either an A - or a B -splitting; see fig. 8. Now each resolved crossing consists of two parallel strands. In the vicinity of each former classical crossing, place one arrow on each of these strands, pointing in opposite directions, figure fig. 9(c). Label these two arrows with a common name and a sign: $+$ if the former crossing has been resolved by an A -splitting and $-$ otherwise. Then pull the state circles apart, untwisting them if needed. The result is the combinatorial representation of the ribbon graph $G_L^{\mathfrak{s}}$, fig. 9(d).



Figure 8: A - and B -splittings

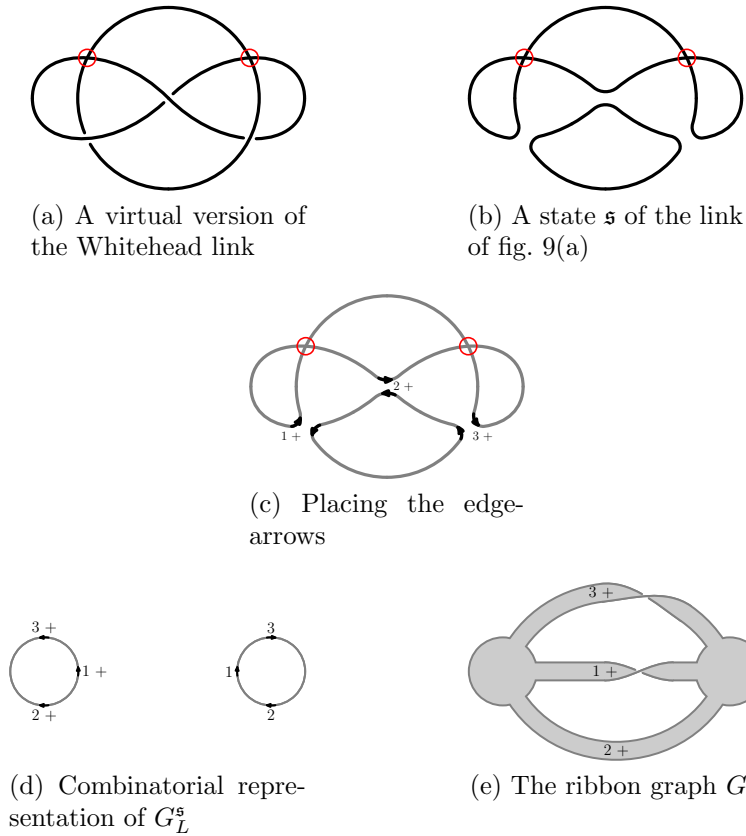


Figure 9: Construction of a $G_L^{\mathfrak{s}}$

The Kauffman bracket and the Bollobás-Riordan polynomial being related by equation (eq. (74)), there is a one-to-one correspondence between the states of a virtual link diagram L and the spanning subribbon graphs of G_L^s . First of all, given the construction of G_L^s , there is a bijection κ between the crossings of L and the edges of G_L^s . Then, writing $C_{s' \neq s}$ for the set of crossings which are resolved differently in s and s' , the bijection between the states of L and the subribbon graphs of G_L^s is

$$\sigma : s' \mapsto \sigma(s') = F \in \mathcal{S}_{G_L^s} \text{ s.t. } E(F) = \kappa(C_{s' \neq s}). \quad (77)$$

Another crucial point, noticed by S. Chmutov [3], is the fact that, given two states s and s' , the ribbon graphs G_L^s and $G_L^{s'}$ are dual to each other with respect to $\kappa(C_{s \neq s'})$:

$$G_L^{s'} = (G_L^s)^{\kappa(C_{s \neq s'})}. \quad (78)$$

This allows us to understand to which state a quasi-tree corresponds. Let us consider a state s' with only one state circle, hereafter called a **connected state**. The ribbon graph $G_L^{s'}$ has only one vertex. But, by equation (eq. (78)), the partial dual of G_L^s with respect to $\kappa(C_{s \neq s'})$ has only one vertex, meaning that the subribbon graph of G_L^s , the edge-set of which is $\kappa(C_{s \neq s'})$, is a quasi-tree. In contrast, a quasi-tree Q defines a unique connected state s' by the equation $E(Q) = \kappa(C_{s \neq s'})$. Then the set of quasi-trees of G_L^s corresponds to the set of connected states of L .

To complete our translation of the expansion (eq. (76)), we now explain to which crossings the orientable live edges correspond. Given a quasi-tree Q of G_L^s , $\mathcal{L}_o(Q)$ is the set of orientable live edges of $(G_L^s)^{E(Q)}$ with respect to F_\emptyset . But there exists a unique connected state $s' = \sigma^{-1}(Q)$ such that $(G_L^s)^{E(Q)} = G_L^{s'}$. The state circle of s' is the boundary of the vertex of $G_L^{s'}$. Then, to determine whether a crossing is live with respect to s' , we mark the resolved crossings in s' , as in the example of fig. 9(c). What we get is a (possibly twisting) circle with $2n$ marks (n = number of crossings of L), labelled with n different names. To decide whether a crossing c is live or not, turn around the state circle of s' , starting at one of the two marks corresponding to c . Before reaching the second mark of c , we meet other labels. A label met twice, called paired, corresponds to an edge in G_L^s which does not cross $\kappa(c)$ in $G_L^{s'}$. In contrast, a label met only once, called single, corresponds to an edge crossing $\kappa(c)$. Then c is live if, from one mark of c to the other, we meet no single lower-ordered label. Otherwise, it is dead.

Finally, a crossing c is orientable with respect to a connected state s' if, from one mark of c to the other, we pass through virtual crossings an even number of times. For example, with respect to the connected state of fig. 10, crossings 1 and 3 are orientable, whereas crossing 2 is non-orientable.

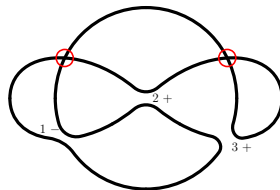


Figure 10: A connected state

To sum up, we have:

- each quasi-tree $Q \in \mathcal{S}_{G_L^s}$ corresponds to a connected state $\mathfrak{s}' = \sigma^{-1}(Q)$,
- $e(G_L^s) = n(L)$ is the number of crossings of L ,
- $e_+((G_L^s)^{E(Q)}) = a_L(\mathfrak{s}')$ is the number of A -splittings of \mathfrak{s}' ,
- $e_-((G_L^s)^{E(Q)}) = b_L(\mathfrak{s}')$ is the number of B -splittings of \mathfrak{s}' ,
- $e_+(\mathcal{L}_o(Q)) =: |\mathcal{L}_o^a(\mathfrak{s}')|$ is the number of live orientable crossings resolved by A -splittings in \mathfrak{s}' ,
- $e_-(\mathcal{L}_o(Q)) =: |\mathcal{L}_o^b(\mathfrak{s}')|$ is the number of live orientable crossings resolved by B -splittings in \mathfrak{s}' .

So we get:

Lemma 7.1 (Connected state expansion) *Let L be a virtual link diagram. For any order for the crossings of L , the Kauffman bracket can be rewritten as*

$$[L](A, B, d) = \sum_{\substack{\text{connected} \\ \text{states } \mathfrak{s}' \text{ of } L}} A^{a_L(\mathfrak{s}')} B^{b_L(\mathfrak{s}')} (1 + Bd/A)^{|\mathcal{L}_o^a(\mathfrak{s}')|} (1 + Ad/B)^{|\mathcal{L}_o^b(\mathfrak{s}')|}. \quad (79)$$

Acknowledgements I thank A. Champanerkar, I. Kofman and N. Stoltzfus for having explained to me some details of their inspiring work [2]. I also acknowledge the anonymous referee's work that led to this improved version.

A Examples

A.1 The Bollobás-Riordan polynomial

We give here an example of our quasi-tree expansion of the signed Bollobás-Riordan polynomial of not necessarily orientable ribbon graphs. We choose the non-orientable signed ribbon graph G of fig. 11. According to equation (eq. (7)), the signed Bollobás-

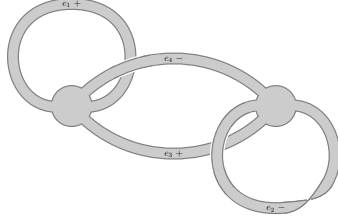


Figure 11: A non-orientable signed ribbon graph

Riordan polynomial of G is

$$R_s(G; x+1, y, z) = 1 + 3y + y^2 + xz + yz + 2xyz + y^2z + xy^2z + xyz^2 + y^2z^2 + xy^2z^3 + x^{-1}y + x^{-1}y^2. \quad (80)$$

We now check that the quasi-tree expansion (eq. (11)) gives the same polynomial. To this aim, according to theorem 5.1, we define

$$P(G; x, y, z) := \sum_{Q \in \mathcal{Q}_G} N(G, Q) S(G_Q), \quad (81a)$$

$$N(G, Q) := (x^{-1/2}y^{1/2})^{e_-(G)} x^{e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q))} y^{n(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)} - e_-(\mathcal{D}(Q) \cup \mathcal{I}_n(Q)))} z^{(k-f+n)(F_{\mathcal{D}(Q) \cup \mathcal{I}_n(Q)})} (1+x)^{e_-(\mathcal{E}_o(Q))} (1+y)^{e_+(\mathcal{E}_o(Q))}, \quad (81b)$$

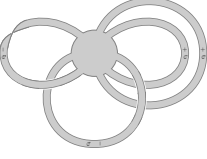
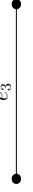
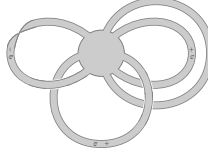

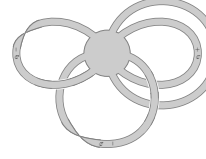
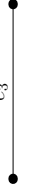
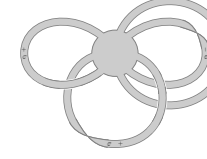

$$S(G_Q) := (x^{1/2}y^{-1/2})^{r(G_Q) + e_-(\mathcal{I}_o(Q))} Ra(G_Q; 1, x^{-1/2}y^{1/2}, x^{1/2}y^{1/2}, x^{1/2}y^{1/2}z^2). \quad (81c)$$

We want to check that $P(G; x, y, z) = R_s(G; x+1, y, z)$. Table 1 lists the information necessary for computing the polynomial P . We get

$$P(G; x, y, z) = (1+x^{-1})y(1+y) + 1 + y + (1+x)y(1+y)z + x(1+y)z + y(1+yz^2) + xyz^2 + xy^2z^3. \quad (82)$$

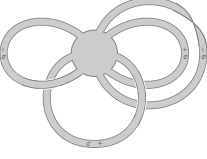
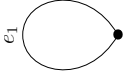
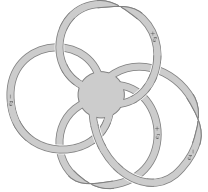

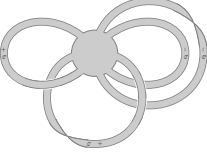

We easily see that the right hand sides of equations (eq. (80)) and (eq. (82)) are equal.

Table 1: Quasi-tree expansion of $R_s(G)$

$E(Q)$	$G^{E(Q)}$	$\mathcal{I}_o(Q), \mathcal{I}_n(Q), \mathcal{D}(Q), \mathcal{E}_o(Q)$	$N(G, Q)$	G_Q	$S(G_Q)$
$\{e_3\}$		$\{e_3\}, \emptyset, \emptyset, \{e_1\}$	$x^{-1/2}y^{1/2}(1+y)$		$x^{1/2}y^{1/2} + x^{-1/2}y^{1/2}$
$\{e_4\}$		$\emptyset, \emptyset, \{e_4\}, \{e_1\}$	$(1+y)$		1
$\{e_2, e_3\}$		$\{e_3\}, \{e_2\}, \emptyset, \{e_1\}$	$x^{1/2}y^{1/2}z(1+y)$		$x^{1/2}y^{1/2} + x^{-1/2}y^{1/2}$
$\{e_2, e_4\}$		$\emptyset, \{e_2\}, \{e_4\}, \{e_1\}$	$xz(1+y)$		1

Continued on next page

Table 1: Quasi-tree expansion of $R_s(G)$

$E(Q)$	$G^{E(Q)}$	$\mathcal{I}_o(Q), \mathcal{I}_n(Q), \mathcal{D}(Q), \mathcal{E}_o(Q)$	$N(G, Q)$	G_Q	$S(G_Q)$
$\{e_1, e_3, e_4\}$		$\{e_1\}, \emptyset, \{e_3, e_4\}, \emptyset$	y		$1 + yz^2$
$\{e_2, e_3, e_4\}$		$\emptyset, \emptyset, \{e_2, e_3, e_4\}, \emptyset$	xyz^2		1
$\{e_1, e_2, e_3, e_4\}$		$\emptyset, \{e_1\}, \{e_2, e_3, e_4\}, \emptyset$	xy^2z^3		1

A.2 The Kauffman bracket

We exemplify here the connected state expansion of the virtual version of the Whitehead link of fig. 9(a). We label the crossings 1, 2 and 3 as in figures fig. 9(c) and fig. 10. We choose the following order: $1 \prec 2 \prec 3$. On one hand,

$$[L](A, B, d) = \sum_{\text{states } \mathfrak{s} \text{ of } L} A^{a_L(\mathfrak{s})} B^{b_L(\mathfrak{s})} d^{c_L(\mathfrak{s})-1} \quad (83)$$

$$= A^3 d + 3A^2 B + 2AB^2 + AB^2 d + B^3. \quad (84)$$

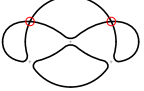
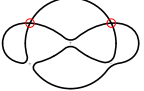
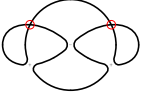
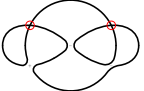
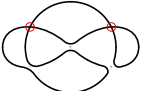
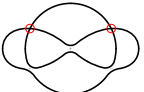
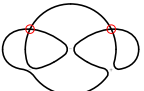
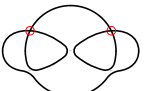
On the other hand,

$$\sum_{\substack{\text{connected} \\ \text{states } \mathfrak{s}' \text{ of } L}} A^{a_L(\mathfrak{s}')} B^{b_L(\mathfrak{s}')} (1 + Bd/A)^{|\mathcal{L}_o^a(\mathfrak{s}')|} (1 + Ad/B)^{|\mathcal{L}_o^b(\mathfrak{s}')|} \quad (85)$$

$$= A^2 B(1 + Bd/A) + A^2 B + AB^2 + A^2 B(1 + Ad/B) + AB^2 + B^3, \quad (86)$$

which is easily checked to be equal to (eq. (84)).

Table 2: States of L

State \mathfrak{s}	$(a_L(\mathfrak{s}), b_L(\mathfrak{s}), c_L(\mathfrak{s}))$	$\mathcal{L}_o^a(\mathfrak{s}), \mathcal{L}_o^b(\mathfrak{s})$ (if $c_L(\mathfrak{s}) = 1$)
	(3, 0, 2)	
	(2, 1, 1)	$\{1\}, \emptyset$
	(2, 1, 1)	\emptyset, \emptyset
	(1, 2, 1)	\emptyset, \emptyset
	(2, 1, 1)	$\emptyset, \{1\}$
	(1, 2, 2)	
	(1, 2, 1)	\emptyset, \emptyset
	(0, 3, 1)	\emptyset, \emptyset

References

- [1] B. Bollobás and O. Riordan. “A polynomial of graphs on surfaces”. *Math. Ann.*, **323**: 81–96, January 2002.
- [2] A. Champanerkar, I. Kofman, and N. Stoltzfus. “Quasi-tree expansion for the Bollobás-Riordan-Tutte polynomial”. May 2007. [arXiv:0705.3458](#).
- [3] S. Chmutov. “Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial”. *J. Combinatorial Theory, Ser. B*, **99** (3):617–638, 2009. [arXiv:0711.3490](#), [doi:10.1016/j.jctb.2008.09.007](#).
- [4] S. Chmutov and I. Pak. “The Kauffman bracket of virtual links and the Bollobás-Riordan polynomial”. *Moscow Math. J.*, **7** (3):409–418, 2007.
- [5] S. Chmutov and J. Voltz. “Thistlethwaite’s theorem for virtual links”. *J. Knot Th. Ram.*, **17** (10):1189–1198, 2008. [arXiv:0704.1310](#).
- [6] O. T. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, and N. W. Stoltzfus. “Alternating sum formulae for the determinant and other link invariants”. November 2006. [arXiv:math/0611025](#).
- [7] O. T. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, and N. W. Stoltzfus. “The Jones polynomial and graphs on surfaces”. *J. Combinatorial Theory, Ser. B*, **98**:384–399, 2008. [arXiv:math.GT/0605571](#).
- [8] E. Dewey. “A quasitree expansion of the Bollobás-Riordan polynomial”. Unpublished work, available at <http://www.math.wisc.edu/~dewey/>.
- [9] C. Godsil and G. Royle. *Algebraic Graph Theory*. Graduate Texts in Mathematics. Springer, 2001.
- [10] S. Huggett and I. Moffatt. “Expansions for the Bollobás-Riordan polynomial of separable ribbon graphs”. *Annals of Combinatorics*, 2011. In press. [arXiv:0710.4266](#).
- [11] L. H. Kauffman. “A Tutte polynomial for signed graphs”. *Discrete Appl. Math.*, **25**: 105–127, 1989.
- [12] I. Moffatt. “Knot invariants and the Bollobás-Riordan polynomial of embedded graphs”. *European Journal of Combinatorics*, **29**:95–107, 2008.
- [13] I. Moffatt. “Partial Duality and Bollobás and Riordan’s Ribbon Graph Polynomial”. *Discrete Mathematics*, **310**:174–183, 2010. [arXiv:0809.3014](#).
- [14] I. Moffatt. “A characterization of partially dual graphs”. *Journal of Graph Theory*, 2011. Published online. [arXiv:0901.1868](#), [doi:10.1002/jgt.20525](#).
- [15] M. Thistlethwaite. “A spanning tree expansion for the jones polynomial”. *Topology*, **26**: 297–309, 1987.
- [16] L. Traldi. “A dichromatic polynomial for weighted graphs and link polynomials”. *Proc. AMS*, **106**:279–286, 1989.

- [17] W. T. Tutte. “A contribution to the theory of chromatic polynomials”. *Canad. J. Math.*, **6**:80–91, 1954.
- [18] F. Vignes-Tourneret. “The multivariate signed Bollobás-Riordan polynomial”. *Discrete Mathematics*, **309**:5968–5981, 2009. [arXiv:0811.1584](#), [doi:10.1016/j.disc.2009.04.026](#).